

# Avoided level crossings in open quantum systems

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## Abstract

At high level density, two states avoid usually crossing at the critical value  $a_{\text{cr}}$  of the parameter  $a$  by which the system is controlled. The wavefunctions of the two states are mixed in a finite parameter range around  $a_{\text{cr}}$ . This holds true for discrete states as well as for narrow resonance states which are coupled via the environment of scattering wavefunctions. We study the influence of avoided level crossings onto four overlapping complex eigenvalues of a symmetric non-Hermitian operator. The mixing of the two wavefunctions around  $a_{\text{cr}}$  is simulated, in each case, by assuming a Gaussian distribution around  $a_{\text{cr}}$ . At high level density, the Gaussian distributions related to avoided crossings of different levels may overlap. Here, new effects arise, especially from the imaginary part of the coupling term via the environment. The results show, moreover, the influence of symmetries onto the multi-level avoided crossing phenomenon.

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## I. INTRODUCTION

Avoided level crossings play an important role in quantum mechanics. They are known for about 80 years [1]. In the case that the Hamilton operator of the system is Hermitian and the real eigenvalues  $E_i$  provide the energies of the discrete states of the system, two of its states never will cross. Instead they avoid crossing when controlled by a parameter. This phenomenon is an observable effect: it consists in an exchange of the two states, including their populations, in a many-particle system. The exchange takes place at the critical value  $a_{\text{cr}}$  of the parameter  $a$ . The corresponding crossing points of the energy trajectories can be found by analytical continuation into the continuum [2].

Narrow resonance states show a similar behavior. They are described well by the complex eigenvalues  $E_i - i/2\Gamma_i$  and biorthogonal eigenfunctions  $\Phi_i$  of a non-Hermitian Hamilton operator [3]. The eigenvalues provide not only the energies  $E_i$  but also the widths  $\Gamma_i$  (inverse lifetimes) of the states. Mostly, the resonance states avoid crossing similar to discrete states. However, in contrast to the eigenvalue trajectories of discrete states, the eigenvalue trajectories of resonance states can cross. The crossing points are called usually *exceptional points* according to Kato [4] who studied first the nontrivial mathematical properties of these singular points. Also in the case of avoided crossings of resonance states, the corresponding crossing points can be found by varying one additional parameter.

Recent studies have shown qualitatively that dynamical phase transitions in open quantum systems occur when the level density is high and, correspondingly, many neighbored states avoid crossing in a relatively small parameter range [3]. As a result, the system is (dynamically) stabilized: the number of states is reduced (by one when the environment consists of one continuum), the individual spectroscopic properties of the original states are lost, and the narrow (trapped) resonance states of the system show collective features. Meanwhile, unexpected experimental results from different fields of physics could be explained qualitatively by means of this phenomenon (see [5] where some of them are sketched). The dynamical stabilization of the system is environmentally induced and appears *only* in open quantum systems which are described by a non-Hermitian Hamilton operator due to the embedding of the system into the environment of scattering wavefunctions.

It is the aim of the present paper to study in detail generic features of open quantum systems at high level density where more than two states avoid crossing and the ratio of

the widths of the states to the energy differences between them is larger than 1. A realistic case of such a type is considered first in nuclear physics [6], then in laser induced continuum structures in atoms [7], and later in many other systems, see the review [3] and also [5]. We restrict the study to four overlapping resonance states and three different avoided level crossings. In difference to the paper [8] on the signatures of symmetric Hamiltonians with three coalescing eigenfunctions, we are interested in the behavior of the *eigenvalues* of a symmetric non-Hermitian operator in the regime of overlapping resonances where the states *avoid crossing*. Here, the wavefunctions of the two states are mixed in a finite parameter range around the critical value  $a_{\text{cr}}$  of the avoided crossing. The mixing range shrinks to one point when analytically continued up to the exceptional point by means of another parameter [9]. At the exceptional point, the two eigenfunctions become linearly dependent from one another [3].

Our calculations are performed with real, complex as well as with imaginary coupling coefficients of the states via the environment. The mixing of the wavefunctions caused by this coupling is simulated, according to the numerical results obtained in [9], by assuming a Gaussian distribution around the critical point of avoided crossing at the critical parameter value  $a_{\text{cr}}$ . At high level density, the Gaussian distributions related to avoided crossings of different levels may overlap. Our results show that, in the regime of overlapping Gaussian distributions, new effects arise from the imaginary part of the coupling term via the environment which are related to width bifurcation. They are much more pronounced than the effects caused by the real part of the coupling term which leads to level repulsion in energy. Furthermore, symmetries between the states are shown to play an important role in the avoided level crossing phenomenon. This result hints at the relation of avoided level crossings to exceptional points.

In section II, the formalism used for the calculations, is formulated. Some results for two crossing levels are given in section III while those for four crossing levels can be found in section IV. The results are summarized and conclusions are drawn in the last section.

## II. FORMALISM

We consider an  $N \times N$  matrix

$$\mathcal{H} = \begin{pmatrix} \varepsilon_1 & \omega_{12} & \dots & \omega_{1N} \\ \omega_{21} & \varepsilon_2 & \dots & \omega_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{N1} & \omega_{N2} & \dots & \varepsilon_N \end{pmatrix} \quad (1)$$

the diagonal elements of which are the  $N$  complex eigenvalues  $\varepsilon_i \equiv e_i - i/2 \gamma_i$  of a non-Hermitian operator. The  $e_i$  and  $\gamma_i$  denote the energies and widths, respectively, of the  $N$  states without account of the interaction of the different states via an environment. This interaction is contained in the values  $\omega_{ik}$  which stand for the coupling matrix elements  $\langle \phi_i | \mathcal{H} | \phi_k \rangle$  of the states  $i$  and  $k$  via the environment, where  $\phi_i$  is the wavefunction of the state  $i$ . The corrections due to the coupling  $\omega_{ii}$  of the state  $i$  to the environment (i.e. to the continuum of scattering wavefunctions into which the system is embedded) lead to the selfenergy of the state  $i$ , see [5]. In atomic physics, these corrections are known as Lamb shift. With the only exception of figure 7, the  $\omega_{ii}$  are assumed, in our model calculations, to be included into the  $\varepsilon_i$ . The  $\omega_{ik}$  are complex, generally. The border case of purely imaginary  $\omega_{ik}$  corresponds to frozen internal degrees of freedom [3]. For discrete states with real energies  $\varepsilon_i = e_i$ , and the  $\omega_{ik}$  are real [3, 5].

The eigenvalues of  $\mathcal{H}$  will be denoted by  $\mathcal{E}_i \equiv E_i - i/2 \Gamma_i$  where  $E_i$  and  $\Gamma_i$  stand for the energy and width, respectively, of the eigenstate  $i$ . The eigenfunctions  $\Phi_i$  of  $\mathcal{H}$  can be represented in the set of basic wavefunctions  $\phi_i$  of the unperturbed matrix (corresponding to the case with vanishing coupling matrix elements  $\omega_{ij}$ ),

$$\Phi_i = \sum_{j=1}^N b_{ij} \phi_j. \quad (2)$$

The  $b_{ij}$  are normalized according to the biorthogonality relations of the wavefunctions  $\{\Phi_i\}$  [3],

$$\langle \Phi_i^* | \Phi_j \rangle = \delta_{ij}. \quad (3)$$

It follows

$$\langle \Phi_i | \Phi_i \rangle = \text{Re} (\langle \Phi_i | \Phi_i \rangle); \quad A_i \equiv \langle \Phi_i | \Phi_i \rangle \geq 1 \quad (4)$$

and

$$\langle \Phi_i | \phi_{j \neq i} \rangle = i \operatorname{Im} (\langle \Phi_i | \phi_{j \neq i} \rangle) = -\langle \Phi_{j \neq i} | \Phi_i \rangle ; \quad |B_i^j| \equiv |\langle \Phi_i | \Phi_{j \neq i} \rangle| \geq 0 . \quad (5)$$

The  $\mathcal{E}_i$  and  $\Phi_i$  contain global features that are caused by many-body forces induced by the coupling  $\omega_{ik}$  of the states via the environment, see [5, 10] and equations (7) and (8) in [3].

Some years ago, the case  $N = 2$  with  $e_i = e_i(a)$ , fixed real  $\omega \equiv \omega_{12} = \omega_{21}$ , and different fixed values of  $\gamma_i$ , including  $\gamma_i = 0$ , is studied as a function of the parameter  $a$  in the neighborhood of avoided and true crossings of the two levels [9]. In this case, the two eigenvalues of  $\mathcal{H}$  are

$$\varepsilon_{i,j} \equiv e_{i,j} - \frac{i}{2} \gamma_{i,j} = \frac{\varepsilon_1 + \varepsilon_2}{2} \pm Z; \quad Z \equiv \frac{1}{2} \sqrt{(\varepsilon_1 - \varepsilon_2)^2 + 4\omega^2} . \quad (6)$$

According to this expression, two interacting discrete states (with  $\gamma_k = 0$ ) avoid always crossing since  $\omega$  and  $\varepsilon_1 - \varepsilon_2$  are real in this case [3]. Resonance states with nonvanishing widths  $\gamma_i$  repel each other in energy according to  $\operatorname{Re}(Z) > 0$  while the widths bifurcate according to  $\operatorname{Im}(Z)$ . The two states cross when  $Z = 0$ .

The results for the  $N = 2$  case [9] show further that the wavefunctions of the two states  $\Phi_1$  and  $\Phi_2$  are mixed in a finite range of the parameter  $a$  around the critical value  $a_{\text{cr}}$  at which the two states avoid crossing. This holds true not only for resonance states but also for discrete states. Furthermore, a nonlinear source term appears in the Schrödinger equation in the neighborhood of an exceptional point and the critical point of an avoided crossing, respectively. This source term causes irreversible processes [3, 5].

In our calculations, the mixing coefficients  $b_{ij}$  (see equation (2)) of the wavefunctions of the two states due to their avoided crossing are not calculated. We simulate the fact that the two wavefunctions are mixed in a finite parameter range around the critical value of their avoid crossing [9] by assuming a Gaussian distribution

$$\omega_{i \neq j} = \omega e^{-(e_i - e_j)^2} \quad (7)$$

for the coupling coefficients.

Calculations for  $N > 2$  and for the case with complex  $\omega$  are not performed up to now. Of special interest is the situation at high level density where the ranges of avoided crossings, defined by (7), of different levels overlap. Some generic results obtained with 2 and 4 resonance states will be presented in the following sections.

### III. $N = 2$ CROSSING LEVELS

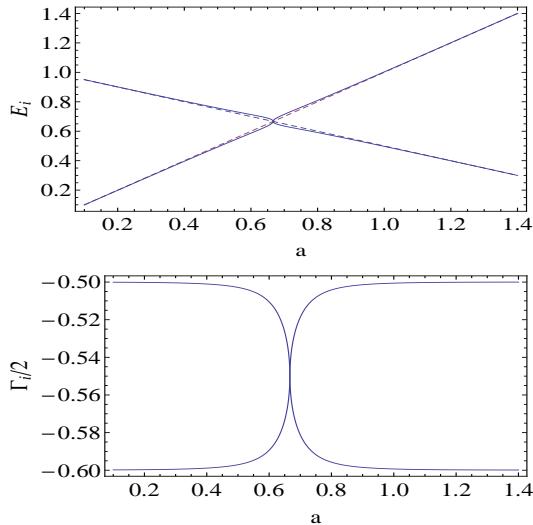


FIG. 1: The energies  $E_i$  (top) and widths  $\Gamma_i/2$  (bottom) for two crossing states as a function of the parameter  $a$  with  $e_1 = 1 - a/2$ ;  $e_2 = a$ ;  $\gamma_1/2 = 0.5$ ;  $\gamma_2/2 = 0.5999$ ;  $\omega = 0.05$ . The dashed lines show the unperturbed  $e_i(a)$ . The exceptional point is at the crossing point of the energies  $e_1$  and  $e_2$ .

First we study the two-level case most properties of which can be found in the literature at different places. Here, we choose the matrix (1) with  $e_i = e_i(a)$ , different fixed values of  $\gamma_i$  and fixed  $\omega \equiv \omega_{12} = \omega_{21}$ . The functional dependence of the energies over the parameter  $a$  is similar as in [9], but the  $\omega$  may be complex. We show the results as a function of the parameter  $a$  in the neighborhood of avoided and true crossings of the two levels.

In figure 1, the energies  $E_i$  and widths  $\Gamma_i$  are shown for two crossing levels with the real coupling coefficient  $\omega = 0.05$ . At the critical value  $a_{\text{cr}}$  of the parameter  $a$ , the two energy trajectories  $e_i(a)$ ;  $i = 1, 2$  cross. Here, also the energy and width trajectories  $E_i(a)$  and  $\Gamma_i(a)$ , respectively, may cross as shown in figure 1. The crossing point of the  $E_i(a)$  and  $\Gamma_i(a)$  is an exceptional point. If one of the input values is slightly different from the values used in figure 1, the  $E_i(a)$  avoid crossing while the  $\Gamma_i(a)$  cross (or the  $E_i(a)$  cross and the widths  $\Gamma_i(a)$  bifurcate). In the first case, the two states are exchanged at the critical point  $a_{\text{cr}}$  of the parameter  $a$ .

The avoided crossing phenomenon of discrete states is known in the literature for many years [1]. As shown in [9], narrow resonance states avoid crossing in energy in a similar

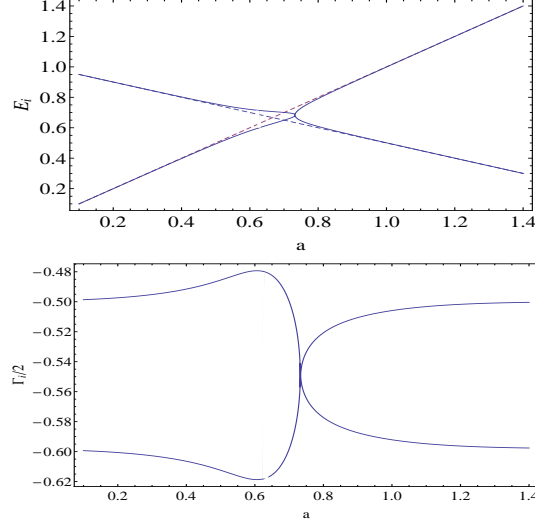


FIG. 2: The energies  $E_i$  (top) and widths  $\Gamma_i/2$  (bottom) for two crossing states as a function of the parameter  $a$  with  $e_1 = 1 - a/2$ ;  $e_2 = a$ ;  $\gamma_1/2 = 0.5$ ;  $\gamma_2/2 = 0.5980$ ;  $\omega = 0.05(1 + i)$ . The dashed lines show the unperturbed  $e_i(a)$ . The exceptional point is shifted in  $a$  relative to the crossing point of the energies  $e_1$  and  $e_2$ .

manner as discrete states. The only difference to the case with discrete states are the nonvanishing widths  $\Gamma_i(a)$  of the resonance states. The widths may cross at the critical parameter value  $a = a_{\text{cr}}$  if  $\omega$  is real and the  $E_i(a)$  avoid crossing, see equation (6).

In figure 2, the situation with two crossing levels but complex coupling  $\omega$  is shown. Also in this case an exceptional point can be found, though for a value of  $\gamma_2$  being different from that in figure 1 due to the other value of  $\omega$ . In figure 2, the position of the exceptional point is shifted in relation to the crossing point of the  $e_i(a)$  trajectories. Furthermore, the difference  $|\Gamma_1 - \Gamma_2|$  blows up around the crossing point of the  $e_i$ . However, at large distances of the parameter  $a$  from the critical value, the widths of the two states approach quickly the values  $\gamma_1$  and  $\gamma_2$ , respectively. If one of the input values is slightly changed, the two eigenvalue trajectories avoid crossing in energy similar as in the case with real coupling  $\omega$  (figure 1) while the  $\Gamma_i(a)$  cross freely, or the  $E_i(a)$  cross and the  $\Gamma_i(a)$  bifurcate. In the first case, the two states are exchanged at the critical parameter value at which the two levels avoid crossing.

The blowing up of the difference  $|\Gamma_1 - \Gamma_2|$  around  $a_{\text{cr}}$  seen in figure 2, is nothing but width bifurcation according to  $\text{Im}(Z)$ , equation (6). This can be seen very clearly in figure

3 calculated with imaginary  $\omega$ . In a relatively small parameter range around the crossing point of the  $e_1$  and  $e_2$  trajectories, we have  $E_1 = E_2$  while the difference  $|\Gamma_1 - \Gamma_2|$  is large. Beyond this parameter range, quickly  $E_i \rightarrow e_j$  and  $\Gamma_i \rightarrow \gamma_j$ . The width bifurcation is caused by  $\text{Im}(\omega)$ . It plays an important role for resonance states that cross at high level density since, in difference to the avoided crossing of discrete states, the coupling of resonance states via the environment is complex in this case [3].

Figures 1 to 3 show the characteristic features of the dynamics of an open quantum system in the neighborhood of an exceptional point: the two crossing states are exchanged at the critical value of the control parameter including their population and, furthermore, the widths bifurcate due to  $\text{Im}(\omega)$ . Population transfer occurs in realistic systems. A few examples are the high-order harmonic generation in a driven two-level atom [11], the ultrafast stimulated Raman parallel adiabatic passage by shaped pulses [12] and the molecular vibrational cooling by adiabatic population transfer from excited to ground vibrational states [13]. Width bifurcation becomes important at high level density as will be shown in the next section.

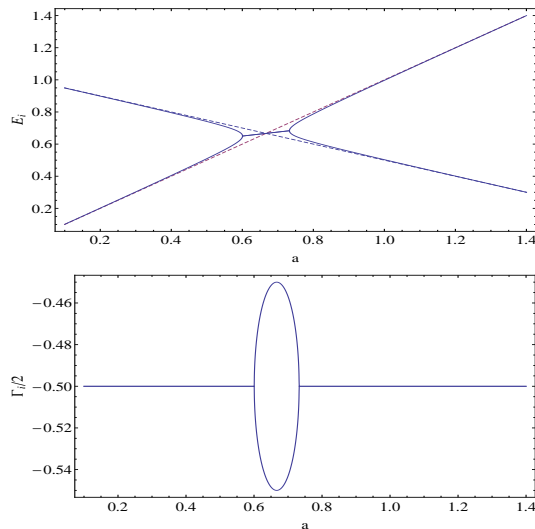


FIG. 3: The energies  $E_i$  (top) and widths  $\Gamma_i/2$  (bottom) for two crossing states as a function of the parameter  $a$  with  $e_1 = 1 - a/2$ ;  $e_2 = a$ ;  $\gamma_1/2 = \gamma_2/2 = 0.5$ ;  $\omega = 0.05i$ . The dashed lines show the unperturbed  $e_i(a)$ . The widths bifurcate in the parameter range where  $E_1 = E_2$ .



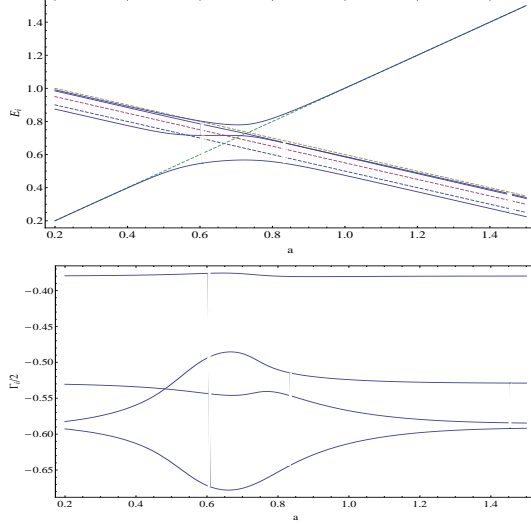


FIG. 4: The energies  $E_i$  (top) and widths  $\Gamma_i/2$  (bottom) for one state crossing three other states as a function of the parameter  $a$  with  $e_1 = 1 - a/2$ ;  $e_2 = 1.05 - a/2$ ;  $e_3 = 1.1 - a/2$ ;  $e_4 = a$ ;  $\gamma_1/2 = 0.5$ ;  $\gamma_2/2 = 0.4$ ;  $\gamma_3/2 = 0.6$ ;  $\gamma_4/2 = 0.58523$ ;  $\omega = 0.05 (1 + i)$ . The dashed lines show the unperturbed  $e_i(a)$ . The widths  $\Gamma_i$  of the three overlapping states 1, 3 and 4 bifurcate (or cross freely) in the parameter range of intersection while energy  $E_2 \approx e_2$  and width  $\Gamma_2 \approx \gamma_2$  remain almost unaffected.

#### IV. N = 4 CROSSING LEVELS

Some results of calculations with one resonance state crossing three other ones are shown in figures 4 up to 9. Mostly we use a linear functional dependence of the energies  $e_i(a)$  on the control parameter  $a$ . Figure 4 shows the role, the symmetry in the overlapping of resonances plays in the critical region of the parameter  $a$ . Energy  $e_2$  and width  $\gamma_2$  of the state 2 are fully symmetric with correspondence to the two states 1 and 3. The avoided crossings of the state 2 with the states 1 and 3, respectively, are disturbed therefore completely symmetrically by the states 3 and 1, respectively. Due to this fact, the state 2 does (almost) not take place in the avoided level crossing phenomenon but appears as an 'observer'. Both, the energy trajectories  $E_2$  and the width trajectories  $\Gamma_2$  follow the trajectories  $e_2$  and  $\gamma_2$  everywhere including the parameter values  $a$  around  $a_{\text{cr}}$ . Such an effect is known from a realistic case with 3 crossing levels in a quantum dot [14].

The crossings and avoided crossings in energy together with the corresponding bifurca-

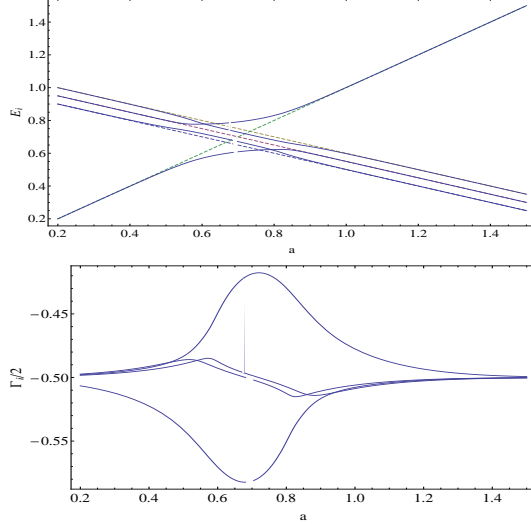


FIG. 5: The same as figure 4 but  $\gamma_1/2 = \gamma_2/2 = \gamma_3/2 = \gamma_4/2 = 0.5$ .  $\omega = 0.05 (1 + i)$ . All states participate in the interaction scenario. Far from the critical parameter range, all  $\Gamma_i$  approach the value 0.5.

tions and crossings in width of the other states 1, 3 and 4 can be seen very clearly in figure 4. Far from the crossing region, all eigenvalue trajectories  $\Gamma_i$ ;  $i = 1, 3, 4$  approach those of the  $\gamma_{i \neq j}$ . An exchange of the states takes place in the critical region around  $a_{\text{cr}}$  with the only exception of state 2.

The numerical symmetry of state 2 relative to the states 1 and 3 is somewhat disturbed when the widths  $\gamma_i$  of all the states are equal to one another, see figure 5. In this case, all four states are affected by the crossings. In the critical region, the avoided level crossings in energy and free crossings in width as well as the free crossings in energy and bifurcations in width can be seen. Far from the critical region, all eigenvalue trajectories  $E_i$  and  $\Gamma_i$  approach the trajectories  $e_j$  and  $\gamma_j$ , respectively. The small shift in energy seen in figure 4 does not appear in figure 5 due to the same values of all  $\gamma_i$  in the last case. This shift is an indicator of the residual influence of the intersection onto the eigenvalues of  $\mathcal{H}$  far from the critical region with avoided level crossings. In other words, it is a signature of the different couplings of the states to the environment in the regime of overlapping.

In difference to the results shown in figures 4 and 5, in those of figure 6 the states 1, 2 and 3 do not interact via the environment:  $\omega_{12} = \omega_{13} = \omega_{23} = 0$ . Their interaction with state 4 via the environment is  $\omega_{i4} = \omega_{4i} = \omega e_i(a) e^{-(e_i - e_4)^2}$ . The results are very similar to

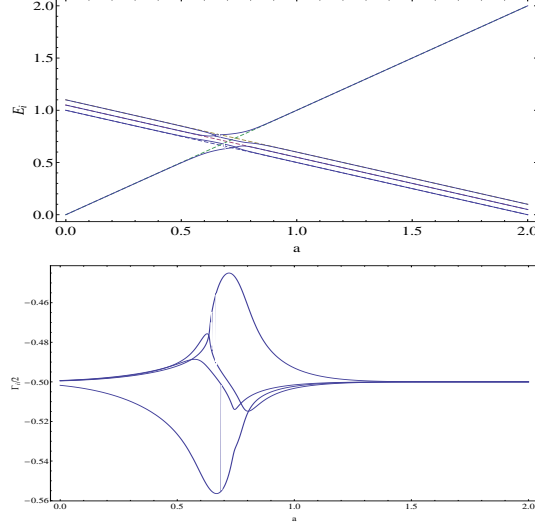


FIG. 6: The same as figure 5 but  $\omega_{i4} = \omega_{4i} = \omega e_i(a) e^{-(e_i - e_4)^2}$ ;  $\omega_{i \neq 4} = \omega_{j \neq 4} = 0$ ;  $\omega_{44} = 0$ .  $\omega = 0.05(1 + i)$ . The widths bifurcate less than in figure 5. All states participate in the intersection scenario. Far from the critical parameter range all  $\Gamma_i$  approach the value 0.5.

those shown in figure 5, except for the fact that the overall interaction is smaller.

Figure 7 shows the influence of the selfenergy term  $\omega_{44} \neq 0$ . It causes some shift in all  $E_i$  and all  $\Gamma_i$  what can be seen best far from the critical region.

The relative influence of the imaginary part of the coupling strength  $\omega$  in the region around  $a_{\text{cr}}$  can be seen in figure 8. The calculations are performed with the same values as in figure 6 but  $\omega = 0.05(1 + i/10)$  instead of  $\omega = 0.05(1 + i)$ . The width bifurcation in figure 8 is smaller than that in figure 6 due to the smaller value of  $\text{Im}(\omega)$ .

In figure 9, we show the results for a completely other distribution of the levels in energy. The functional dependence of the energies is chosen according to a Coulomb-like potential. All the features discussed in figures 4 to 8 can be seen also with this distribution. Even the influence of the symmetry of the states 1 and 3 in relation to the state 2 causes the state 2 to be an 'observer', in the same manner as discussed in figure 4. Further results obtained in other calculations with the  $e_i$  distribution of figure 9 are not shown in the present paper.

Similar results are obtained also for the case that two resonance states cross two other ones according to, e.g.,  $e_1 = 1 - a/2$ ;  $e_2 = 1.05 - a/2$ ;  $e_3 = 0.05 + a$ ;  $e_4 = a$ , and different values for the  $\gamma_i$ . Numerical results have shown furthermore that the Gaussian distribution (7) used in the calculations, ensures the overlapping of the different avoided crossings, i.e.

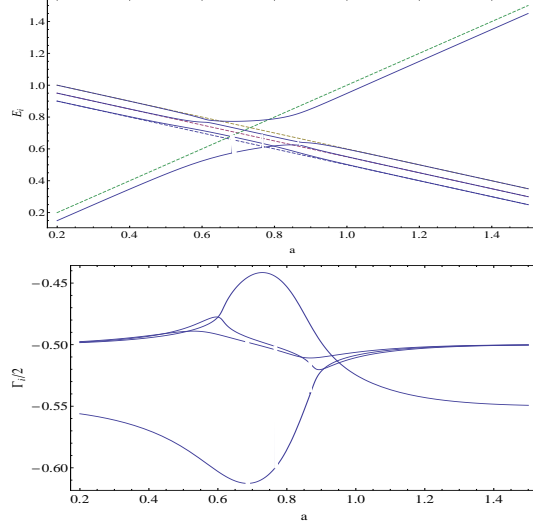


FIG. 7: The same as figure 6 but  $\omega_{44} = \omega = 0.05(1 + i)$ . The widths  $\Gamma_i$  bifurcate in the parameter range of intersection. The selfenergy term  $\omega_{44} \neq 0$  causes shift in energy and width far from the critical parameter range.

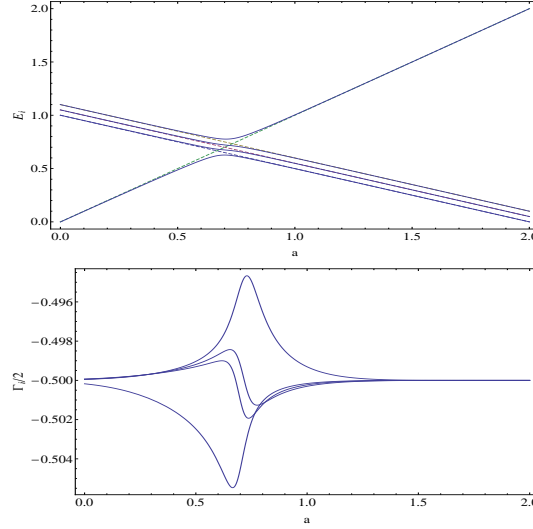


FIG. 8: The same as figure 6 but  $\omega = 0.05 (1 + i/10)$ . All states participate in the avoided level crossings. Far from the critical parameter range all  $\Gamma_i$  approach the value 0.5.

the overlapping of the parameter ranges in which the wavefunctions of the two crossing states are mixed. Results with a broader Gaussian distribution are almost the same as those shown in the figures of the present paper.

The features caused by the true and avoided crossings shown in the figures 4 to 9 are

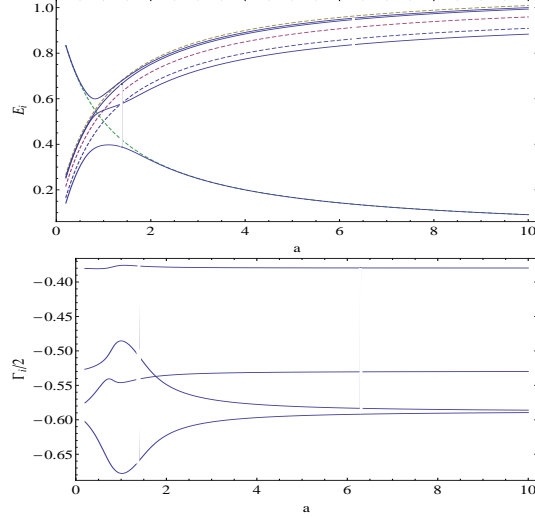


FIG. 9: The energies  $E_i$  (top) and widths  $\Gamma_i/2$  (bottom) for four crossing states as a function of the parameter  $a$  with  $e_1 = 1 - \frac{1}{a+1}$ ;  $e_2 = 1.05 - \frac{1}{a+1}$ ;  $e_3 = 1.1 - \frac{1}{a+1}$ ;  $e_4 = \frac{1}{a+1}$ ;  $\gamma_1/2 = 0.5$ ;  $\gamma_2/2 = 0.4$ ;  $\gamma_3/2 = 0.6$ ;  $\gamma_4/2 = 0.58523$ ;  $\omega = 0.05 (1 + i)$ . The dashed lines show the unperturbed  $e_i(a)$ . The widths  $\Gamma_1$ ,  $\Gamma_3$ ,  $\Gamma_4$  bifurcate in the critical parameter range while the width  $\Gamma_2 \approx \gamma_2$  and the energy  $E_2 \approx e_2$  remain almost unchanged (as in figure 4).

generic. In any case, the influence of the level crossings occurring in a relatively small region around the critical value  $a_{\text{cr}}$ , onto the four eigenvalues of  $\mathcal{H}$  is, generally, large. Only at large distances from the critical region, the widths  $\Gamma_i$  of the states approach the values  $\gamma_j$  with  $j \neq i$  in most cases. A shift occurs due to the selfenergy term. In the critical region, the eigenvalue trajectories are exchanged usually. Exceptions occur when the exchange is prevented due to certain symmetries, such that one of the states acts as an 'observer'.

The results obtained at large distances from the critical region are true from a mathematical point of view. They are, however, hardly realized in physical systems for the following reason. In the critical region, the system is split into different parts due to width bifurcation: one part contains the long-lived states while another one contains the short-lived states. The short-lived states decay quickly, and the long-lived states form a new system the wavefunctions of which are strongly mixed. This process occurring in the critical region, is irreversible. As a consequence, the system will never reach the parameter region far from the critical value  $a_{\text{cr}}$ . Instead, a dynamical phase transition takes place as discussed in [3, 5]. The figures 4 to 9 show that signatures of the dynamical phase transition can be seen already

in the case of only four overlapping resonances.

## V. CONCLUSIONS

In the present paper, we have shown numerical results for avoided crossings of two and four states in a relatively large range of the control parameter  $a$  by using the matrix (1). The avoided crossing of two levels at the critical value  $a_{\text{cr}}$  influences the complex eigenvalues of the non-Hermitian operator  $\mathcal{H}$  in a small parameter range around  $a_{\text{cr}}$ . Here, the levels avoid crossing in energy while the widths cross freely or they cross freely in energy while the widths bifurcate. In the first case, the two states are exchanged, including their populations. The results can be understood by means of the eigenvalue equations (6).

The influence of more than one avoided level crossing occurring in a small critical parameter range, onto the eigenvalues of  $\mathcal{H}$  is stronger than in the case with only one avoided level crossing in the critical region. The influence can be seen still beyond the critical region. The imaginary part of the interaction  $\omega$  of the states via the environment affects the eigenvalues stronger than the real part of  $\omega$  does. Especially width bifurcation appearing at nearby avoided level crossings is usually strong. The reason for these results are the nonlinear source terms that appear around an avoided level crossing in the Schrödinger equation [3, 5]. They become more important when several avoided level crossings are near to one another and the wavefunctions of the different states overlap. An exception from these results occurs when one state is related symmetrically to two neighboring states. In such a case, the state does (almost) not participate in the avoided level crossing phenomenon, but is solely an 'observer' of the avoided crossing of the two neighbored states. This result corresponds to sensitivity of exceptional points to the distortion of their symmetry, see the discussion in appendix E in [5].

In physical systems  $\omega$  as well as  $\gamma_i$  are usually functions of energy. Furthermore, the ratio  $\text{Im}(\omega) / \text{Re}(\omega)$  increases with energy. We did not simulate these effects in our numerical calculations since we are interested in the study of generic effects. In any case, the critical range of nearby avoided crossings affects the eigenstates of a non-Hermitian operator in a relatively large parameter range. Far from the critical region with overlapping resonances, the eigenvalue trajectories  $\mathcal{E}_i = E_i - i/2 \Gamma_i$  approach, as it must be from a mathematical point of view, the trajectories  $\varepsilon_j = e_j - i/2 \gamma_j$  with  $i \neq j$  as a rule. The selfenergy term

causes some shift of the  $\mathcal{E}_i$  relative to the  $\varepsilon_j$ .

The results shown in figures 4 to 9 are relevant for physical systems at high level density where different avoided crossings overlap. This range is characterized by level repulsion and width bifurcation as the results of the present paper show. Width bifurcation limits the existence of the system: it causes a splitting of the system into different parts with significant different lifetimes. Finally, a dynamical phase transition takes place [3, 5]. Further studies with more than four overlapping resonance states and with  $\omega = \omega(a)$  are in progress. Furthermore, we will apply the results of multi-level avoided crossing also to the case of atomic and semiconductor cavity QED which is studied in [15–17].

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